

The Schmidt number and partially entanglement breaking channels in infinite dimensions*

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Abstract

A definition of the Schmidt number of a state of an infinite dimensional bipartite quantum system is given and properties of the corresponding family of Schmidt classes are considered. The existence of states with a given Schmidt number such that any their countable convex decomposition does not contain pure states with finite Schmidt rank is established.

Partially entanglement breaking channels in infinite dimensions are studied. Several properties of these channels well known in finite dimensions are generalized to the infinite dimensional case. At the same time, the existence of partially entanglement breaking channels (in particular, entanglement breaking channels) such that all operators in any their Kraus representations have infinite rank is proved.

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1 Introduction

The Schmidt rank of a pure state and its extension to mixed states called the Schmidt number are important quantitative characteristics of entanglement in bipartite quantum systems.

The Schmidt rank of a pure state of bipartite system AB determined by a unit vector $|\psi\rangle$ is defined as the number of nonzero terms in the Schmidt decomposition

$$|\psi\rangle = \sum_i \lambda_i |\alpha_i\rangle \otimes |\beta_i\rangle$$

of this vector, it coincides with the rank of the reduced states.

The Schmidt number of a mixed state ω of a finite dimensional bipartite system AB is defined in [18] as the minimum over all its decompositions into convex combination of pure states of the maximal Schmidt rank of these pure states (see Section 3). In [18] it is shown that the Schmidt number does not increase under LOCC-operations and that the set of states with the Schmidt number not exceeding k (the Schmidt class of order k) can be characterized in terms of k -positive maps. In the subsequent papers [12, 16, 17] different properties of the Schmidt number and of the family of the Schmidt classes

are considered. In particular, the notion of k -Schmidt witnesses generalizing the notion of entanglement witnesses is introduced and analyzed.

By using the Schmidt number the notion of partially entanglement breaking quantum channels is introduced in [3]. It turns out that this notion is closely related to the necessary condition of nondecreasing of the Holevo quantity of an ensemble of quantum states under action of a quantum channel [14, Theorem 1].

This paper is devoted to infinite-dimensional generalizations of the above concepts. It is partially motivated by author's intension to extend the above-mentioned result in [14] to "continuous variable" systems and channels.

In Section 3 a natural generalization of the Schmidt number to states of infinite dimensional bipartite quantum systems is considered. Since existence of non-countably decomposable separable states (see [5]) makes the finite dimensional formula for the Schmidt number non-adequate, the "continuous" modification of this formula (based on the notion of the essential supremum of a function with respect to a given measure) is proposed. It is shown that this formula gives a reasonable definition of the Schmidt number in the sense that the corresponding Schmidt classes (the sets of states with the Schmidt number $\leq k$) coincide with the convex closures of the sets of pure states with the Schmidt rank $\leq k$.

Properties of the Schmidt classes in infinite dimensions are considered in Section 4. In particular, the characterization the Schmidt class of order k in terms of k -positive maps (generalizing Theorem 1 in [18]) is given. It is shown that an arbitrary state in the Schmidt class of order k can be represented as a barycenter of a probability measure supported by pure states with the Schmidt rank $\leq k$. At the same time, the existence of states with a given Schmidt number such that any their countable convex decomposition does not contain pure states with a finite Schmidt rank is established.

A definition and some properties of partially entanglement breaking channels in infinite dimensions are considered in Section 5. In contrast to the finite dimensional case, the class of k -partially entanglement breaking channels does not coincide with the class of channels having the Kraus representation consisting of operators of rank $\leq k$ (the latter is a proper subclass of the former). Moreover, the existence of partially entanglement breaking channels (in particular, of entanglement breaking channels) such that all operators in any their Kraus representations have infinite rank is proved.

2 Basic notations

We will use the following notations:

$\mathcal{H}, \mathcal{H}', \mathcal{K}$ – separable Hilbert spaces;

$\mathfrak{B}(\mathcal{H})$ – the Banach space of all bounded operators in \mathcal{H} ;

$\mathfrak{T}(\mathcal{H})$ – the Banach space of all trace-class operators in \mathcal{H} ;

$\mathfrak{T}_+(\mathcal{H})$ – the cone of all positive trace-class operators in \mathcal{H} ;

$\mathfrak{S}(\mathcal{H})$ – the subset of $\mathfrak{T}_+(\mathcal{H})$ consisting of operators with a unit trace.

The unit operator in a Hilbert space \mathcal{H} and the identity transformation of the space $\mathfrak{T}(\mathcal{H})$ are denoted $I_{\mathcal{H}}$ and $\text{Id}_{\mathcal{H}}$ correspondingly.

Operators in $\mathfrak{S}(\mathcal{H})$ are denoted $\rho, \sigma, \omega, \dots$ and called density operators or states since each density operator uniquely defines a normal state on $\mathfrak{B}(\mathcal{H})$.

We denote by $\text{cl}(\mathcal{A})$, $\text{co}(\mathcal{A})$, $\overline{\text{co}}(\mathcal{A})$ and $\text{extr}(\mathcal{A})$ the closure, the convex hull, the convex closure and the set of all extreme points of a subset \mathcal{A} of a linear topological space correspondingly [9, 1, 11].

The set of all Borel probability measures on a closed subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ endowed with the topology of weak convergence is denoted $\mathcal{P}(\mathcal{A})$ [2, 10]. This set can be considered as a complete separable metric space [10]. The barycenter $\mathbf{b}(\mu)$ of a measure μ in $\mathcal{P}(\mathcal{A})$ is the state in $\overline{\text{co}}(\mathcal{A})$ defined by the Bochner integral

$$\mathbf{b}(\mu) = \int_{\mathcal{A}} \rho \mu(d\rho).$$

For an arbitrary subset $\mathcal{B} \subseteq \overline{\text{co}}(\mathcal{A})$ let $\mathcal{P}_{\mathcal{B}}(\mathcal{A})$ be the subset of $\mathcal{P}(\mathcal{A})$ consisting of all measures with the barycenter in \mathcal{B} .

A finite or infinite collection of states $\{\rho_i\} \subset \mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ with the corresponding probability distribution $\{\pi_i\}$ is conventionally called *ensemble* and denoted $\{\pi_i, \rho_i\}$. An ensemble can be considered as an atomic (discrete) measure in $\mathcal{P}(\mathcal{A})$. The barycenter of this measure is the average state $\sum_i \pi_i \rho_i$ of the corresponding ensemble.

A positive trace-preserving linear map $\Phi : \mathfrak{T}(\mathcal{H}) \rightarrow \mathfrak{T}(\mathcal{H}')$ such that the dual map $\Phi^* : \mathfrak{B}(\mathcal{H}') \rightarrow \mathfrak{B}(\mathcal{H})$ is completely positive is called *quantum channel* [4, 8]. The set of all quantum channels from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ is denoted $\mathfrak{F}(\mathcal{H}, \mathcal{H}')$.

3 The Schmidt number

Let \mathcal{H} and \mathcal{K} be separable Hilbert spaces. The Schmidt rank $\text{SR}(\omega)$ of a pure state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ can be defined as the rank of the isomorphic states $\text{Tr}_{\mathcal{K}}\omega$ and $\text{Tr}_{\mathcal{H}}\omega$.

If the spaces \mathcal{H} and \mathcal{K} are finite dimensional then the Schmidt number of an arbitrary state ω in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ is defined in [18] as follows

$$\text{SN}(\omega) = \inf_{\sum_i \pi_i \omega_i = \omega} \sup_i \text{SR}(\omega_i) \quad (1)$$

(the infimum is over all ensembles $\{\pi_i, \omega_i\}$ of pure states with the average state ω). By using the Caratheodory theorem it is easy to show that for each natural k the set $\mathfrak{S}_k(\mathcal{H} \otimes \mathcal{K}) = \{\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \mid \text{SN}(\omega) \leq k\}$ is compact and coincides with the convex hull of all pure states having the Schmidt rank $\leq k$. This implies that the function $\omega \mapsto \text{SN}(\omega)$ is lower semicontinuous on the set $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$. Thus we have the increasing finite sequence¹

$$\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \mathfrak{S}_3 \subset \dots \subset \mathfrak{S}_{n-1} \subset \mathfrak{S}_n = \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$$

of compact sets, where \mathfrak{S}_1 is the set of separable (non-entangled) states and $n = \min\{\dim \mathcal{H}, \dim \mathcal{K}\}$.

If the spaces \mathcal{H} and \mathcal{K} are infinite dimensional then the right side of (1) is well defined but can not be used as an adequate definition of the Schmidt number. This follows from existence of separable states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ (called non-countably decomposable), which can not be decomposed into a countable convex combination of product pure states [5]. The nonexistence of a decomposition into product pure states shows that the right side of (1) is > 1 for any such separable state ω contradicting to the natural requirement for the Schmidt number.²

We will show below that a reasonable generalization of definition (1) to the infinite dimensional case is given by the following formula

$$\text{SN}(\omega) = \inf_{\mu \in \mathcal{P}_{\{\omega\}}(\text{extr} \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}))} \text{ess sup}_{\mu} \text{SR}(\cdot) \quad (2)$$

¹Here and in what follows we will often write \mathfrak{S}_k instead of $\mathfrak{S}_k(\mathcal{H} \otimes \mathcal{K})$ for brevity.

²This problem is similar to the problem arising in infinite dimensional generalization of the convex roof construction of entanglement monotones: the existence of non-countably decomposable separable states makes the discrete version of this construction not adequate (see Remark 9 in [13]).

where "ess sup $_{\mu}$ " means the essential supremum with respect to the measure μ [1, Section 13.1]. Note that $\text{ess sup}_{\mu} \text{SR}(\cdot) = \|\text{SR}\|_{\infty}$ – the norm of the function SR in the space $L^{\infty}(X, \mu)$, where $X = \text{extr} \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$.

Proposition 1. A) *The function $\text{SN}(\omega)$ defined by (2) is lower semicontinuous on $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$. For each state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ the infimum in (2) is achieved at some measure in $\text{extr} \mathcal{P}_{\{\omega\}}(\text{extr} \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}))$.*

B) *For each natural k the set $\mathfrak{S}_k(\mathcal{H} \otimes \mathcal{K}) = \{\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \mid \text{SN}(\omega) \leq k\}$, where $\text{SN}(\omega)$ is defined by (2), is closed and convex. It coincides with the convex closure of all pure states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ having the Schmidt rank $\leq k$.*

C) *If ω is a finite rank state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ then the value $\text{SN}(\omega)$ defined by (2) coincides with the value $\text{SN}(\omega)$ defined by (1).*

Proof. Since the nonnegative function $\omega \mapsto \text{SR}(\omega)$ is lower semicontinuous on $\text{extr} \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, the first assertion of the proposition follows from Proposition 10 in the Appendix.

The second assertion follows from the first one and Lemma 1 in [5].

To prove the third assertion assume that the right side of (2) is equal to $k < +\infty$. Equality of the right side of (1) to k follows from coincidence of the convex hull and the convex closure of the closed subset

$$\mathfrak{S}_k^{\omega} = \{\varpi \in \text{extr} \mathfrak{S}(\text{supp} \omega) \mid \text{SR}(\varpi) \leq k\}$$

of the finite-dimensional space $\mathfrak{T}(\text{supp} \omega)$ (see [1, Corollary 5.33]). \square

The following proposition generalizes Proposition 1 in [18] to the infinite dimensional case.

Proposition 2. *The Schmidt number of a state of an infinite-dimensional bipartite system (defined by (2)) does not increase under LOCC-operations.*

This proposition can be reduced to Proposition 1 in [18] by using the following approximation result.

Lemma 1. *Let $\{P_n\}$ and $\{Q_n\}$ be increasing sequences of finite rank projectors strongly converging respectively to $I_{\mathcal{H}}$ and to $I_{\mathcal{K}}$. For an arbitrary state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ let $\omega_n = (\text{Tr} P_n \otimes Q_n \cdot \omega)^{-1} P_n \otimes Q_n \cdot \omega \cdot P_n \otimes Q_n$. Then*

$$\lim_{n \rightarrow +\infty} \text{SN}(\omega_n) = \text{SN}(\omega).$$

If $\text{SN}(\omega) < +\infty$ then there exists n_0 such that $\text{SN}(\omega_n) = \text{SN}(\omega)$ for all $n \geq n_0$.

Proof. By lower semicontinuity of the Schmidt number (Proposition 1A) it suffices to show that

$$\text{SN}(\omega_n) \leq \text{SN}(\omega), \quad \forall n. \quad (3)$$

Since the state ω belongs to the convex closure of the set $\mathfrak{S}_{\text{SN}(\omega)}^p$ of pure states with the Schmidt rank $\leq \text{SN}(\omega)$ (Proposition 1B), there exists a sequence $\{\omega_m\}$ from the convex hull of the set $\mathfrak{S}_{\text{SN}(\omega)}^p$ converging to the state ω such that $\lim_{m \rightarrow +\infty} \text{SN}(\omega_m) = \text{SN}(\omega)$. For each m inequality (3) with $\omega = \omega_m$ is directly verified. By lower semicontinuity of the Schmidt number passing to the limit as $m \rightarrow +\infty$ implies (3). \square

4 Some properties of the Schmidt classes \mathfrak{S}_k

If $\dim \mathcal{H} = \dim \mathcal{K} = +\infty$ we have the increasing infinite sequence

$$\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \mathfrak{S}_3 \subset \dots \subset \mathfrak{S}_{n-1} \subset \mathfrak{S}_n \subset \dots$$

of closed convex subsets of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, where \mathfrak{S}_1 is the set of separable (nonentangled) states.

Let \mathfrak{S}_k^p be the closed subset of \mathfrak{S}_k consisting of pure states.

Proposition 3. A) *An arbitrary state in \mathfrak{S}_k can be represented as a barycenter of some measure in $\text{extr}\mathcal{P}(\mathfrak{S}_k^p)$.*

B) *There exist states ω in $\mathfrak{S}_k \setminus \mathfrak{S}_{k-1}$ such that the operator $\omega - \lambda\sigma$ is not positive for any $\lambda > 0$ and any pure state σ with a finite Schmidt rank.³ For any such state ω we have*

$$\omega = \sum_i \pi_i \omega_i, \quad \{\omega_i\} \subset \text{extr}\mathfrak{S}(\mathcal{H} \otimes \mathcal{K}), \quad \Rightarrow \quad \text{SR}(\omega_i) = +\infty \quad \forall i.$$

C) *An arbitrary pure state in $\mathfrak{S}_k \setminus \mathfrak{S}_{k-1}$ can be approximated by states in $\mathfrak{S}_k \setminus \mathfrak{S}_{k-1}$ having the property stated in B).*

Proof. The first assertion directly follows from Proposition 1, the second one – from the example in the Appendix 6.2 (after Proposition 11).

The third assertion can be derived from the construction in the example in the Appendix 6.2 by noting that functions with non-vanishing Fourier coefficients form a dense subset in $L^2([0, 2\pi))$ and that an arbitrary set $\{|\psi_i\rangle\}_{i=1}^k$

³It is assumed that $\mathfrak{S}_0 = \emptyset$, so that $\mathfrak{S}_1 \setminus \mathfrak{S}_0 = \mathfrak{S}_1$ is the set of separable states.

of orthogonal unit vectors in a separable Hilbert space \mathcal{H} can be represented as an image of the set $\{|\varphi_i\rangle \otimes |i\rangle\}_{i=1}^k \subset L^2([0, 2\pi)) \otimes \mathcal{K}$ under some unitary map from $L^2([0, 2\pi)) \otimes \mathcal{K}$ onto \mathcal{H} , where $\{|i\rangle\}_{i=1}^k$ is an orthonormal basis in the space \mathcal{K} . \square

Consider the characterization of the set \mathfrak{S}_k in term of k -positive maps (generalizing Theorem 1 in [18] to the infinite-dimensional case).

Proposition 4. *A state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ belongs to the set \mathfrak{S}_k if and only if the operator $\Lambda_k \otimes \text{Id}_{\mathcal{K}}(\omega)$ is positive for any k -positive linear transformation Λ_k of the space $\mathfrak{T}(\mathcal{H})$.*

Proof. Let $\omega_0 \in \mathfrak{S}_k$. By Proposition 3 there exists a measure μ_0 in $\mathcal{P}(\mathfrak{S}_k^p)$ such that $\omega_0 = \int \omega \mu_0(d\omega)$. Since $\Lambda_k \otimes \text{Id}_{\mathcal{K}}(\omega) \geq 0$ for any $\omega \in \mathfrak{S}_k^p$ by definition of k -positivity, $\Lambda_k \otimes \text{Id}_{\mathcal{K}}(\omega_0) = \int \Lambda_k \otimes \text{Id}_{\mathcal{K}}(\omega) \mu_0(d\omega) \geq 0$.

The converse assertion can be derived from the corresponding finite dimensional result ([18, Theorem 1]) by using the approximation based on Lemma 1.

Let $\omega_0 \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \setminus \mathfrak{S}_k$, i.e. $\text{SN}(\omega_0) > k$. By Lemma 1 there exist projectors $P \in \mathfrak{B}(\mathcal{H})$ and $Q \in \mathfrak{B}(\mathcal{K})$ of the same finite rank such that the state $\omega_* = (\text{Tr} P \otimes Q \cdot \omega_0)^{-1} P \otimes Q \cdot \omega_0 \cdot P \otimes Q$ does not belong to the set \mathfrak{S}_k . Let $\mathcal{H}_* = P(\mathcal{H})$ and $\mathcal{K}_* = Q(\mathcal{K})$. By Theorem 1 in [18] there exists a k -positive map $\Lambda_k : \mathfrak{T}(\mathcal{H}_*) \rightarrow \mathfrak{T}(\mathcal{H}_*)$ such that the operator $\Lambda_k \otimes \text{Id}_{\mathcal{K}_*}(\omega_*)$ is not positive. Consider the k -positive map $\Lambda_k \circ \Pi$, where $\Pi(\cdot) = P(\cdot)P$. Then the operator $(\Lambda_k \circ \Pi) \otimes \text{Id}_{\mathcal{K}}(\omega_0)$ is not positive, since otherwise the operator

$$I_{\mathcal{H}} \otimes Q \cdot (\Lambda_k \circ \Pi) \otimes \text{Id}_{\mathcal{K}}(\omega_0) \cdot I_{\mathcal{H}} \otimes Q = (\text{Tr} P \otimes Q \cdot \omega_0) \Lambda_k \otimes \text{Id}_{\mathcal{K}_*}(\omega_*)$$

is positive in contradiction to the choice of Λ_k . \square

By using the compactness criterion for subsets of $\mathfrak{T}_+(\mathcal{H} \otimes \mathcal{K})$ (see the Proposition in the Appendix in [6]) one can generalize Proposition 1 in [12] to infinite dimensions.

Proposition 5. *An arbitrary state $\omega_k \in \mathfrak{S}_k$ can be represented as follows*

$$\omega_k = (1 - p)\omega_{k-1} + p\delta, \quad p \in [0, 1], \quad (4)$$

where $\omega_{k-1} \in \mathfrak{S}_{k-1}$ and δ is a state having the Schmidt number $\geq k$ such that the operator $\delta - \lambda\sigma$ is not positive for any $\lambda > 0$ and any $\sigma \in \mathfrak{S}_{k-1}$.

Among all such decompositions there is a decomposition with minimal p .

A state having the property of the state δ is called *k-edge* state in [12]. In contrast to the finite dimensional case, to prove that δ is a *k-edge* state it is not sufficient to show that $\delta - \lambda\sigma$ is not positive for any $\lambda > 0$ and any $\sigma \in \mathfrak{S}_{k-1}^p$. This follows from Proposition 3B.

Proof. Let $\mathcal{M} = \{0\} \cup \{A \in \mathfrak{T}_+(\mathcal{H} \otimes \mathcal{K}) \mid A \leq \omega_k, (\text{Tr} A)^{-1} A \in \mathfrak{S}_{k-1}\}$ be a closed subset of the cone $\mathfrak{T}_+(\mathcal{H} \otimes \mathcal{K})$.

Assume $\mathcal{M} \neq \{0\}$. By the above-mentioned compactness criterion for subsets of $\mathfrak{T}_+(\mathcal{H} \otimes \mathcal{K})$ the set \mathcal{M} is compact. Hence there exists $A_0 \in \mathcal{M}$ such that $\text{Tr} A_0 = \sup_{A \in \mathcal{M}} \text{Tr} A$. Denoting $p = 1 - \text{Tr} A_0$, $\omega_{k-1} = (\text{Tr} A_0)^{-1} A_0$ and $\delta = p^{-1}(\omega_k - A_0)$ we obtain (4) with minimal p .

If $\mathcal{M} = \{0\}$ then the only way to obtain (4) is to take $p = 1$ and $\delta = \omega_k$.

□

Since the family $\{\mathfrak{S}_k\}$ consists of closed convex subsets of the Banach space $\mathfrak{T}_h(\mathcal{H} \otimes \mathcal{K})$ of all trace class Hermitian operators in $\mathcal{H} \otimes \mathcal{K}$, for each k and each $\omega_0 \in \mathfrak{S}_k \setminus \mathfrak{S}_{k-1}$ the Hahn-Banach theorem implies existence of a Hermitian operator W such that $\text{Tr} W \omega \geq 0$ for all $\omega \in \mathfrak{S}_{k-1}$ and $\text{Tr} W \omega_0 < 0$. This operator W is called *k-Schmidt witness (k-SW) detecting ω_0* [12].

An explicit form of *k-SW* detecting a state δ such that the operator $\delta - \lambda\sigma$ is not positive for any $\lambda > 0$ and $\sigma \in \mathfrak{S}_{k-1}$ (in particular, the *k-edge* state δ in decomposition (4)) is given in [12, Lemma 1] (in finite dimensions). This is the operator

$$W = P - \frac{\varepsilon}{c} C,$$

where P is an arbitrary positive operator whose range coincides with the kernel of the state δ , C is an arbitrary positive operator such that $\text{Tr} C \delta > 0$, $\varepsilon = \inf_{|\varphi\rangle \langle \varphi| \in \mathfrak{S}_{k-1}^p} \langle \varphi | P | \varphi \rangle$ and $c = \|C\|$ (compactness arguments imply $\varepsilon > 0$).

To apply the above construction in infinite dimensions we have to impose additional conditions ensuring positivity of ε .

Proposition 6. *The above construction of k-SW detecting the k-edge state δ is valid in infinite dimensions if the state δ has finite rank and the spectrum of the operator P does not contain zero.*

Proof. By Proposition 3A to prove this assertion it suffices to show that $\varepsilon = \inf_{|\varphi\rangle \langle \varphi| \in \mathfrak{S}_{k-1}^p} \langle \varphi | P | \varphi \rangle > 0$ if P is the projector on the kernel of δ .

Assume that $\varepsilon = 0$. Then there exists a sequence $\{|\varphi_n\rangle\}$ of vectors such that $|\varphi_n\rangle \langle \varphi_n| \in \mathfrak{S}_{k-1}^p$ for each n and $\lim_{n \rightarrow +\infty} \langle \varphi_n | I_{\mathcal{H} \otimes \mathcal{K}} - P | \varphi_n \rangle = 1$. The weak compactness of the unit ball of $\mathcal{H} \otimes \mathcal{K}$ implies existence of a subsequence

$\{|\varphi_{n_k}\rangle\}$ weakly converging to a vector $|\varphi_*\rangle$. Since $I_{\mathcal{H}\otimes\mathcal{K}} - P$ is a finite rank projector, we have

$$\langle\varphi_*|I_{\mathcal{H}\otimes\mathcal{K}} - P|\varphi_*\rangle = \lim_{n\rightarrow+\infty} \langle\varphi_n|I_{\mathcal{H}\otimes\mathcal{K}} - P|\varphi_n\rangle = 1.$$

Thus $|\varphi_*\rangle$ is a unit vector belonging to the range of δ and hence the subsequence $\{|\varphi_{n_k}\rangle\}$ converges to the vector $|\varphi_*\rangle$ in the norm topology. This implies $|\varphi_*\rangle\langle\varphi_*| \in \mathfrak{S}_{k-1}^p$ contradicting to the basic property of the state δ . \square

5 Partially entanglement breaking channels

The notion of a k -partially entanglement-breaking (k -PEB) channel in finite dimensions is introduced in [3] as a natural generalization of the notion of an entanglement-breaking channel (which is 1-PEB). According to [3], *a channel $\Phi : \mathfrak{T}(\mathcal{H}) \rightarrow \mathfrak{T}(\mathcal{H}')$ is called k -partially entanglement-breaking if for an arbitrary Hilbert space \mathcal{K} the Schmidt number of the state $\Phi \otimes \text{Id}_{\mathcal{K}}(\omega) \in \mathfrak{S}(\mathcal{H}' \otimes \mathcal{K})$ does not exceed k for any state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$.*

By using the definition of the Schmidt number introduced in Section 3 the above definition of k -partially entanglement breaking channels is directly generalized to the infinite-dimensional case.

Denote by $\mathfrak{P}_k(\mathcal{H}, \mathcal{H}')$ the class of k -partially entanglement breaking channels from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$. Since the set $\mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K})$ is closed and convex, $\mathfrak{P}_k(\mathcal{H}, \mathcal{H}')$ is a closed convex subset of the set $\mathfrak{F}(\mathcal{H}, \mathcal{H}')$ of all channels from $\mathfrak{T}(\mathcal{H})$ to $\mathfrak{T}(\mathcal{H}')$ endowed with the strong convergence topology [6].

Proposition 4 implies the following characterization of k -PEB channels (generalizing the corresponding finite-dimensional results [3, 7]).

Proposition 7. *A channel Φ is k -partially entanglement breaking if and only if the map $\Lambda_k \circ \Phi$ is completely positive for any k -positive map Λ_k .*

By definition $\Phi \in \mathfrak{P}_k(\mathcal{H}, \mathcal{H}')$ means that $\Phi \otimes \text{Id}_{\mathcal{K}}(\omega) \in \mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K})$ for any $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$. By the following proposition it suffices to verify the above inclusion only for one pure state.

Proposition 8. *Let $\Phi : \mathfrak{T}(\mathcal{H}) \rightarrow \mathfrak{T}(\mathcal{H}')$ be a quantum channel. If there exists a pure state $|\psi\rangle\langle\psi|$ in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ having full rank partial states $\text{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi| \cong \text{Tr}_{\mathcal{H}}|\psi\rangle\langle\psi|$ such that $\Phi \otimes \text{Id}_{\mathcal{K}}(|\psi\rangle\langle\psi|) \in \mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K})$ then the channel Φ is k -partially entanglement-breaking.*

Proof. Let $|\psi\rangle = \sum_{i=1}^{+\infty} \mu_i |i\rangle \otimes |i\rangle$, where $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H} \cong \mathcal{K}$ and $\mu_i > 0$ for all i . Let $P_n = \sum_{i=1}^n |i\rangle\langle i|$ be a projector in $\mathfrak{B}(\mathcal{K})$.

By Proposition 2 we have

$$\Phi \otimes \text{Id}_{\mathcal{K}}(|\psi_n\rangle\langle\psi_n|) = c_n I_{\mathcal{H}} \otimes P_n \cdot \Phi \otimes \text{Id}_{\mathcal{K}}(|\psi\rangle\langle\psi|) \cdot I_{\mathcal{H}} \otimes P_n \in \mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K})$$

where $|\psi_n\rangle = c_n \sum_{i=1}^n \mu_i |i\rangle \otimes |i\rangle$ and $c_n = [\sum_{i=1}^n \mu_i^2]^{-1/2}$.

Let $\mathcal{H}_n = \text{lin}(\{|i\rangle\}_{i=1}^n)$ and $\mathcal{K}_n = \text{lin}(\{|i\rangle\}_{i=1}^n)$ be n -dimensional subspaces of \mathcal{H} and \mathcal{K} . An arbitrary vector $|\varphi\rangle$ in $\mathcal{H}_n \otimes \mathcal{K}_n$ can be represented as follows $|\varphi\rangle = \sum_{i,j=1}^n \gamma_{ij} |i\rangle \otimes |j\rangle = \sum_{i=1}^n \mu_i |i\rangle \otimes A|i\rangle$, where $A = \sum_{i,j=1}^n (\mu_i)^{-1} \gamma_{ij} |j\rangle\langle i|$ is an operator in $\mathfrak{B}(\mathcal{K}_n)$. Thus $|\varphi\rangle\langle\varphi| = I_{\mathcal{H}_n} \otimes A \cdot |\psi_n\rangle\langle\psi_n| \cdot I_{\mathcal{H}_n} \otimes A^*$ and hence

$$\Phi \otimes \text{Id}_{\mathcal{K}}(|\varphi\rangle\langle\varphi|) = I_{\mathcal{H}} \otimes A \cdot \Phi \otimes \text{Id}_{\mathcal{K}}(|\psi_n\rangle\langle\psi_n|) \cdot I_{\mathcal{H}} \otimes A^* \in \mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K}).$$

This means that the restriction of the channel Φ to the set $\mathfrak{S}(\mathcal{H}_n)$ is k -partially entanglement breaking. By the below Lemma 2 the channel Φ is k -partially entanglement breaking. \square

The following lemma reduces a proof of the k -partially entanglement breaking property of a channel $\Phi : \mathfrak{T}(\mathcal{H}) \rightarrow \mathfrak{T}(\mathcal{H}')$ to analysis of its finite-dimensional restrictions.

Lemma 2. *Let $\{\mathcal{H}_n\}$ be an increasing sequence of subspaces of \mathcal{H} such that $\text{cl}(\bigcup_n \mathcal{H}_n) = \mathcal{H}$. If the restriction of the channel Φ to the set $\mathfrak{S}(\mathcal{H}_n)$ is k -partially entanglement breaking for each n then the channel Φ is k -partially entanglement breaking.*

Proof. Since an arbitrary state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ can be approximated by a sequence $\{\omega_n\}$ such that $\text{supp Tr}_{\mathcal{K}} \omega_n \subset \mathcal{H}_n$ (see Lemma 1), this assertion follows from closedness of the set $\mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K})$. \square

Let $|\psi\rangle\langle\psi|$ be a pure state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ having full rank partial states $\text{Tr}_{\mathcal{K}} |\psi\rangle\langle\psi| \cong \text{Tr}_{\mathcal{H}} |\psi\rangle\langle\psi| = \sigma$. Consider the Choi-Jamiołkowski one-to-one correspondence

$$\mathfrak{F}(\mathcal{H}, \mathcal{H}') \ni \Phi \leftrightarrow \Phi \otimes \text{Id}_{\mathcal{K}}(|\psi\rangle\langle\psi|) \in \mathfrak{C}_{\sigma} \doteq \{\omega \in \mathfrak{S}(\mathcal{H}' \otimes \mathcal{K}) \mid \text{Tr}_{\mathcal{H}'} \omega = \sigma\},$$

which is a topological isomorphism provided the set $\mathfrak{F}(\mathcal{H}, \mathcal{H}')$ of all channels is endowed with the strong convergence topology [6, Proposition 3]. Proposition 8 implies the following observation.

Corollary 1. *The restriction of the Choi-Jamiolkowski isomorphism to the class $\mathfrak{P}_k(\mathcal{H}, \mathcal{H}')$ is an isomorphism between this class and the closed subset $\mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K}) \cap \mathfrak{C}_\sigma$ of the set $\mathfrak{S}(\mathcal{H}' \otimes \mathcal{K})$.*

The set $\mathfrak{P}_k(\mathcal{H}, \mathcal{H}') \setminus \mathfrak{P}_{k-1}(\mathcal{H}, \mathcal{H}')$ corresponds to the set

$$(\mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K}) \setminus \mathfrak{S}_{k-1}(\mathcal{H}' \otimes \mathcal{K})) \cap \mathfrak{C}_\sigma, \quad k = 2, 3, \dots$$

under this isomorphism.

In [3] it is proved that a channel Φ is k -PEB if and only if it has the Kraus representation

$$\Phi(\cdot) = \sum_i V_i(\cdot) V_i^* \quad (5)$$

such that $\text{rank} V_i \leq k$ for all i (this is a natural generalization of the well known characterization of entanglement breaking finite-dimensional channels proved in [7]). In infinite dimensions the class of k -PEB channels is wider than the class of channels having the last property.

Proposition 9. A) *A channel Φ belongs to the class $\mathfrak{P}_k(\mathcal{H}, \mathcal{H}')$ if it has the Kraus representation (5) such that $\text{rank} V_i \leq k$ for all i .*

B) *There exist channels Φ in $\mathfrak{P}_k(\mathcal{H}, \mathcal{H}') \setminus \mathfrak{P}_{k-1}(\mathcal{H}, \mathcal{H}')$ with the following property⁴*

$$\Phi(\cdot) = \sum_i V_i(\cdot) V_i^* \quad \Rightarrow \quad \text{rank} V_i = +\infty \quad \forall i.$$

Proof. The first assertion is obvious, since for an arbitrary pure state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ the expression $\Phi \otimes \text{Id}_\mathcal{K}(\omega) = \sum_i V_i \otimes I_\mathcal{K} \cdot \omega \cdot V_i^* \otimes I_\mathcal{K}$ gives a decomposition of the state $\Phi \otimes \text{Id}_\mathcal{K}(\omega)$ into a convex combination of pure states with the Schmidt rank $\leq k$.

To prove the second one consider a state ω in $\mathfrak{S}_k(\mathcal{H}' \otimes \mathcal{K}) \setminus \mathfrak{S}_{k-1}(\mathcal{H}' \otimes \mathcal{K})$ having the property stated in Proposition 3B. We may assume that $\text{Tr}_{\mathcal{H}'} \omega$ is a full rank state in $\mathfrak{S}(\mathcal{K})$. Let $|\psi\rangle\langle\psi|$ be a purification of this state in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$. By Corollary 1 the channel Φ_ω corresponding to the state ω via the Choi-Jamiolkowski isomorphism induced by the state $|\psi\rangle\langle\psi|$ belongs to the set $\mathfrak{P}_k(\mathcal{H}, \mathcal{H}') \setminus \mathfrak{P}_{k-1}(\mathcal{H}, \mathcal{H}')$. If we assume that $\Phi_\omega(\cdot) = \sum_i V_i(\cdot) V_i^*$ with $\text{rank} V_{i_0} < +\infty$ for some i_0 then we will obtain a contradiction to the property

⁴It is assumed that $\mathfrak{P}_0(\mathcal{H}, \mathcal{H}') = \emptyset$, so that $\mathfrak{P}_1(\mathcal{H}, \mathcal{H}') \setminus \mathfrak{P}_0(\mathcal{H}, \mathcal{H}') = \mathfrak{P}_1(\mathcal{H}, \mathcal{H}')$ is the class of entanglement breaking channels.

of the state ω , since $V_{i_0} \otimes \text{Id}_{\mathcal{K}}|\psi\rangle \neq 0$ (otherwise $V_{i_0}(\text{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi|)(V_{i_0})^* = 0$ contradicting to a full rank of the state $\text{Tr}_{\mathcal{K}}|\psi\rangle\langle\psi|$). \square

Corollary 2. *There exist entanglement breaking channels such that all operators in any their Kraus representations have infinite rank.*

6 Appendix

6.1 One property of the set $\mathfrak{S}(\mathcal{H})$.

We consider here a corollary of the compactness criterion for subsets of probability measures on the set $\mathfrak{S}(\mathcal{H})$ (described in detail in [13, Section 1]), which states that *a subset \mathcal{P} of $\mathcal{P}(\mathfrak{S}(\mathcal{H}))$ is compact (in the weak convergence topology) if and only if $\{\mathbf{b}(\mu) \mid \mu \in \mathcal{P}\}$ is a compact subset of $\mathfrak{S}(\mathcal{H})$.*

Proposition 10. *Let f be a nonnegative lower-semicontinuous function on a closed subset \mathcal{A} of $\mathfrak{S}(\mathcal{H})$. The function*

$$F(\rho) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}(\mathcal{A})} \text{ess sup}_{\mu} f(\cdot) \quad (6)$$

is lower semicontinuous on the set $\overline{\text{co}}(\mathcal{A})$.⁵ For each state $\rho \in \overline{\text{co}}(\mathcal{A})$ the infimum in (6) is achieved at some measure in $\text{extr}\mathcal{P}_{\{\rho\}}(\mathcal{A})$.

For each $c \geq 0$ the set $\{\rho \in \overline{\text{co}}(\mathcal{A}) \mid F(\rho) \leq c\}$ coincides with the convex closure of the set $\{\rho \in \mathcal{A} \mid f(\rho) \leq c\}$.

Proof. Note first that the function $F(\rho)$ is well defined on the set $\overline{\text{co}}(\mathcal{A})$ by Lemma 1 in [5].

Show that the functional

$$\mathcal{P}(\mathcal{A}) \ni \mu \mapsto \hat{f}(\mu) = \text{ess sup}_{\mu} f(\cdot) \quad (7)$$

is concave and lower semicontinuous. Since for a given measure $\mu \in \mathcal{P}(\mathcal{A})$ the μ -essential supremum of the function f (coinciding with the norm $\|f\|_{\infty}$ in the space $L^{\infty}(\mathcal{A}, \mu)$) is the least upper bound of the increasing family of the norms $\|f\|_p$ in the space $L^p(\mathcal{A}, \mu)$, $p \in [1, +\infty)$, concavity and lower semicontinuity of functional (7) follow from concavity and lower semicontinuity of the functional

$$\mathcal{P}(\mathcal{A}) \ni \mu \mapsto \|f\|_p = \sqrt[p]{\int_{\mathcal{A}} [f(\rho)]^p \mu(d\rho)}$$

⁵” ess sup_{μ} ” means the essential supremum with respect to the measure μ [1, Sec.13.1].

(lower semicontinuity of this functional follows from the basic properties of the weak convergence topology, see [2, Chapter I, Sec.2]).

By concavity and lower semicontinuity of functional (7) and compactness of the set $\mathcal{P}_{\{\rho\}}(\mathcal{A})$ (provided by the above-stated compactness criterion) the infimum in the definition of the value $F(\rho)$ for each ρ in $\overline{\text{co}}(\mathcal{A})$ is achieved at a particular measure in $\text{extr}\mathcal{P}_{\{\rho\}}(\mathcal{A})$.

Suppose function (6) is not lower semicontinuous. Then there exists a sequence $\{\rho_n\} \subset \overline{\text{co}}(\mathcal{A})$ converging to a state $\rho_0 \in \overline{\text{co}}(\mathcal{A})$ such that

$$\exists \lim_{n \rightarrow +\infty} F(\rho_n) < F(\rho_0). \quad (8)$$

As proved before for each $n = 1, 2, \dots$ there exists a measure μ_n in $\mathcal{P}_{\{\rho_n\}}(\mathcal{A})$ such that $F(\rho_n) = \hat{f}(\mu_n)$. Since the sequence $\{\rho_n\}$ is a compact set, the above-stated compactness criterion implies existence of a subsequence $\{\mu_{n_k}\}$ converging to a particular measure μ_0 . By continuity of the map $\mu \mapsto \mathbf{b}(\mu)$ the measure μ_0 belongs to the set $\mathcal{P}_{\{\rho_0\}}(\mathcal{A})$. Lower semicontinuity of functional (7) implies

$$F(\rho_0) \leq \hat{f}(\mu_0) \leq \liminf_{k \rightarrow +\infty} \hat{f}(\mu_{n_k}) = \lim_{k \rightarrow +\infty} F(\rho_{n_k}),$$

contradicting to (8).

The last assertion of the proposition follows from the previous ones and Lemma 1 in [5]. \square

6.2 Existence of a state with a given finite Schmidt number such that any its countable convex decomposition does not contain pure states with a finite Schmidt rank

In this section we show first that the separable non-countably decomposable state constructed in [5] has, in fact, more stronger property: any countable decomposition of this state does not contain pure states with finite Schmidt rank (non-countable decomposability means nonexistence of such decomposition into product pure states – states with the Schmidt rank = 1). By using this observation we construct a state with a given finite Schmidt number such that any countable decomposition of this state does not contain pure states with a finite Schmidt rank.

We present the construction of the above mentioned separable state following the notations of [5]. Consider the one-dimensional rotation group represented as the interval $[0, 2\pi)$ with addition mod 2π . Let $\mathcal{H} = L^2([0, 2\pi))$ with the normalized Lebesgue measure $\frac{dx}{2\pi}$, and let $\{|k\rangle; k \in \mathbf{Z}\}$ be the orthonormal basis of trigonometric functions, so that

$$\langle k|\psi\rangle = \int_0^{2\pi} e^{-ikx} \psi(x) \frac{dx}{2\pi}.$$

Consider the unitary representation $x \rightarrow V_x$, where $V_x = \sum_{-\infty}^{+\infty} e^{ikx} |k\rangle\langle k|$, so that $(V_u \psi)(x) = \psi(x + u)$.

For arbitrary unit vectors $|\varphi_j\rangle \in \mathcal{H}_j \simeq L^2([0, 2\pi))$, $j = 1, 2$, consider the separable state

$$\rho_{12} = \int_0^{2\pi} V_x^{(1)} |\varphi_1\rangle\langle\varphi_1| (V_x^{(1)})^* \otimes V_x^{(2)} |\varphi_2\rangle\langle\varphi_2| (V_x^{(2)})^* \frac{dx}{2\pi}. \quad (9)$$

The following proposition strengthens the assertion of Theorem 3 in [5]. It is obtained by a natural generalization of the proof of this theorem.

Proposition 11. *Let ρ_{12} be the separable state defined in (9). If the vectors $|\varphi_j\rangle$ have nonvanishing Fourier coefficients then the operator*

$$\rho_{12} - \lambda \sigma$$

is not positive for any $\lambda > 0$ and any pure state σ with finite Schmidt rank.

It follows that any countable decomposition of the state ρ_{12} does not contain pure states with finite Schmidt rank.

Proof. Suppose there exists a vector $|\psi\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with the Schmidt rank n such that

$$\rho_{12} \geq |\psi\rangle\langle\psi|. \quad (10)$$

Let $|\psi\rangle = \sum_{i=1}^n |\alpha_i^1\rangle \otimes |\alpha_i^2\rangle$, where $\{|\alpha_i^j\rangle\}_{i=1}^n$, $j = 1, 2$, are sets of orthogonal vectors. Inequality (10) implies

$$\int_0^{2\pi} |\langle\lambda_1|V_x^{(1)}|\varphi_1\rangle|^2 |\langle\lambda_2|V_x^{(2)}|\varphi_2\rangle|^2 \frac{dx}{2\pi} \geq \left| \sum_{i=1}^n \langle\lambda_1|\alpha_i^1\rangle \langle\lambda_2|\alpha_i^2\rangle \right|^2 \quad (11)$$

for arbitrary $\lambda_j \in L^2([0, 2\pi))$, $j = 1, 2$.

Consider the linear maps

$$L^2([0, 2\pi)) \ni \lambda \mapsto \Phi_j(\lambda) = \{\langle \alpha_i^j | \lambda \rangle\}_{i=1}^n \in \mathbb{C}^n$$

and

$$L^2([0, 2\pi)) \ni \lambda \mapsto \Psi_j(\lambda) = \overline{\langle \lambda | V_x^{(j)} | \varphi_j \rangle} = \sum_{k=-\infty}^{+\infty} \langle \varphi_j | k \rangle \langle k | \lambda \rangle e^{-ikx}, \quad j = 1, 2.$$

Let \mathcal{H}_0 be a dense subset of $L^2([0, 2\pi))$ consisting of functions having finite number of nonzero Fourier coefficients (trigonometric polynomials). Since $\langle \varphi_j | k \rangle \neq 0$ for all k , the maps Ψ_j , $j = 1, 2$, are linear isomorphisms from \mathcal{H}_0 onto itself. Hence (11) implies

$$|\langle A_1(\xi), \Xi(A_2(\eta)) \rangle_{\mathbb{C}^n}|^2 \leq \int_0^{2\pi} |\xi(x)\eta(x)|^2 \frac{dx}{2\pi}, \quad \xi, \eta \in \mathcal{H}_0, \quad (12)$$

where $A_j(\cdot) = \Phi_j(\Psi_j^{-1}(\cdot))$, $j = 1, 2$, are linear maps from \mathcal{H}_0 to \mathbb{C}^n and Ξ is the complex conjugation in \mathbb{C}^n .

Since $\{\Phi_2(\lambda) \mid \lambda \in L^2([0, 2\pi))\} = \mathbb{C}^n$, we have $\{\Phi_2(\lambda) \mid \lambda \in \mathcal{H}_0\} = \mathbb{C}^n$ and hence $\{A_2(\xi) \mid \xi \in \mathcal{H}_0\} = \mathbb{C}^n$. Thus there exists a subset $|\eta_1\rangle, \dots, |\eta_n\rangle$ of the basis $\{|k\rangle\}$ such that the vectors $A_2(\eta_1), \dots, A_2(\eta_n)$ form a basis in \mathbb{C}^n . Since $|\eta_i(x)| = 1$, (12) implies

$$|\langle A_1(\xi), \Xi(A_2(\eta_i)) \rangle_{\mathbb{C}^n}|^2 \leq \int_0^{2\pi} |\xi(x)|^2 \frac{dx}{2\pi} = \|\xi\|^2, \quad i = \overline{1, n}, \quad \xi \in \mathcal{H}_0.$$

Hence the map A_1 is bounded on \mathcal{H}_0 and can be extended to the bounded linear operator A_1 from $L^2([0, 2\pi))$ to \mathbb{C}^n .

By the similar reasoning the map A_2 can be extended to the bounded linear operator A_2 from $L^2([0, 2\pi))$ to \mathbb{C}^n .

Since the anti-linear operator $B = A_1^* \circ \Xi \circ A_2$ in the space $L^2([0, 2\pi))$ has rank $\leq n$, it can be represented as follows $B(\cdot) = \sum_{i=1}^n \langle \cdot | \beta_i^2 \rangle |\beta_i^1\rangle$, where $\{|\beta_i^j\rangle\}$, $j = 1, 2$, are sets of vectors in $L^2([0, 2\pi))$ and the set $\{|\beta_i^1\rangle\}$ consists of linearly independent vectors.

Thus we can rewrite (12) as follows

$$\left| \sum_{i=1}^n \langle \xi | \beta_i^1 \rangle \langle \eta | \beta_i^2 \rangle \right|^2 \leq \int_0^{2\pi} |\xi(x)\eta(x)|^2 \frac{dx}{2\pi}. \quad (13)$$

By Lemma 3 below for arbitrary $\varepsilon > 0$ one can find a subset $\mathcal{A} \subset [0, 2\pi)$ with the Lebesgue measure $< \varepsilon$ such that the functions $\beta_1^1, \beta_2^1, \dots, \beta_n^1$ are linearly independent on \mathcal{A} . Thus for each i we can find a function ξ supported by \mathcal{A} such that $\langle \xi | \beta_i^1 \rangle \neq 0$ but $\langle \xi | \beta_j^1 \rangle = 0$ for all $j \neq i$. Hence for this function ξ and arbitrary function η supported by the complement of \mathcal{A} , the right hand side of (13) vanishes implying $\langle \eta | \beta_i^2 \rangle = 0$ and therefore β_i^2 vanishes a.e. on the complement of \mathcal{A} . It follows that the support of β_i^2 has measure $\leq \varepsilon$, i.e. β_i^2 vanishes a.e. Thus we have $B = 0$. This shows that $|\psi\rangle = 0$. \square

In the following lemma the linear independence of measurable functions f_1, \dots, f_n on a measurable subset $\mathcal{A} \subset \mathbb{R}$ means that any nontrivial linear combination of these functions $\neq 0$ a.e. on \mathcal{A} .

Lemma 3. *Let f_1, \dots, f_n be linearly independent measurable functions on $[a, b]$. For arbitrary $\varepsilon > 0$ there exists a subset $\mathcal{A} \subset [a, b]$ with the Lebesgue measure $\mu(\mathcal{A}) < \varepsilon$ such that the functions f_1, \dots, f_n are linearly independent on \mathcal{A} .*

Proof. For $n = 1, 2$ the assertion of the lemma is obvious. Assume it is valid for given n and show its validity for $n + 1$.⁶

By the assumption for arbitrary $\varepsilon > 0$ and for each family $\{f_i\} \setminus f_j$, $j = \overline{1, n+1}$, there exists a subset $\mathcal{A}_j^\varepsilon \subset [a, b]$ with $\mu(\mathcal{A}_j^\varepsilon) < \varepsilon$ such that the functions of the above family are linearly independent on $\mathcal{A}_j^\varepsilon$.

If the assertion of the lemma is not valid for $n + 1$ then there exists $\varepsilon_* > 0$ such that the functions f_1, \dots, f_{n+1} are linearly dependent on any subset $\mathcal{A} \subset [a, b]$ with $\mu(\mathcal{A}) < \varepsilon_*$.

Let $\varepsilon < \varepsilon_*/2(n+1)$ and $\mathcal{A}^\varepsilon = \bigcup_{j=1}^{n+1} \mathcal{A}_j^\varepsilon$. Choose a finite collection $\{\mathcal{B}_k\}$ of disjoint subsets of $[a, b] \setminus \mathcal{A}^\varepsilon$ such that $\mu(\mathcal{B}_k) < \varepsilon_*/2$ and $\bigcup_k \mathcal{B}_k = [a, b] \setminus \mathcal{A}^\varepsilon$.

For each k let $\mathcal{C}_k = \mathcal{A}^\varepsilon \cup \mathcal{B}_k$. Since $\mu(\mathcal{C}_k) < \varepsilon_*$ there exists a set $\{\lambda_i^k\}_{i=1}^{n+1}$ of complex numbers such that

$$\sum_{i=1}^{n+1} \lambda_i^k f_i(x) = 0 \quad \text{a.e. on } \mathcal{C}_k, \quad \sum_{i=1}^{n+1} |\lambda_i^k| > 0. \quad (14)$$

Since $\mathcal{A}_j^\varepsilon \subset \mathcal{C}_k$ for all j , it is easy to see that $\lambda_i^k \neq 0$ for all i . So, we may assume that $\lambda_{n+1}^k = 1$. Since the functions f_1, \dots, f_{n+1} are linearly independent on $[a, b] = \bigcup_k \mathcal{C}_k$, there exist k_1 and k_2 such that $\{\lambda_i^{k_1}\}_{i=1}^{n+1} \neq$

⁶There is an elegant proof of this lemma non-using the induction method [15].

$\{\lambda_i^{k_2}\}_{i=1}^{n+1}$. It follows from (14) that

$$\sum_{i=1}^n (\lambda_i^{k_1} - \lambda_i^{k_2}) f_i(x) = 0 \quad \text{a.e. on } \mathcal{A}_{n+1}^\varepsilon \subseteq \mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}, \quad \sum_{i=1}^n |\lambda_i^{k_1} - \lambda_i^{k_2}| > 0$$

contradicting to the construction of the set $\mathcal{A}_{n+1}^\varepsilon$. \square

Example of a state ω with $\text{SN}(\omega) = k \in \mathbb{N}$ such that the operator $\omega - \lambda\sigma$ is not positive for any pure state σ with a finite Schmidt rank and any $\lambda > 0$.

Let $\{|\varphi_1^i\rangle\}_{i=1}^k$ and $\{|\varphi_2^i\rangle\}_{i=1}^k$ are collections of orthogonal unit vectors in $\mathcal{H}_1 = L^2([0, 2\pi])$ and $\mathcal{H}_2 = L^2([0, 2\pi])$ correspondingly with non-vanishing Fourier coefficients. Let \mathcal{K} be the k -dimensional Hilbert space with the orthonormal basis $\{|i\rangle\}_{i=1}^k$. For each natural n consider the state

$$\rho_{123}^n = \int_0^{2\pi/n} V_x^{(1)} \otimes V_x^{(2)} \otimes I_{\mathcal{K}} \cdot |\Omega\rangle\langle\Omega| \cdot (V_x^{(1)})^* \otimes (V_x^{(2)})^* \otimes I_{\mathcal{K}} \frac{ndx}{2\pi} \quad (15)$$

in $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$, where $|\Omega\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^k |\varphi_1^i\rangle \otimes |\varphi_2^i\rangle \otimes |i\rangle$ is a unit vector in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K}$.

In what follows (speaking about the Schmidt rank and the Schmidt number) we will treat the space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K}$ as the tensor product of the spaces \mathcal{H}_1 and $\mathcal{H}_2 \otimes \mathcal{K}$.

Since the state ρ_{123}^n belongs to the convex closure of the family of local unitary translations of the state $|\Omega\rangle\langle\Omega|$ such that $\text{SR}(|\Omega\rangle\langle\Omega|) = k$, we have $\text{SN}(\rho_{123}^n) \leq k$ for all n . Since the sequence $\{\rho_{123}^n\}$ tends to the state $|\Omega\rangle\langle\Omega|$, this and lower semicontinuity of the Schmidt number (Proposition 1A) show that $\text{SN}(\rho_{123}^n) = k$ for sufficiently large n .

Suppose that

$$\rho_{123}^n \geq \lambda |\Psi\rangle\langle\Psi|, \quad (16)$$

for some $\lambda > 0$, where $|\Psi\rangle\langle\Psi|$ is a pure state in $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K})$ with a finite Schmidt rank. Let $P_i = I_{\mathcal{H}_1} \otimes I_{\mathcal{H}_2} \otimes |i\rangle\langle i|$. Since $\sum_{i=1}^k P_i = I_{\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{K}}$, there exists i_0 such that $P_{i_0} |\Psi\rangle \neq 0$. We may assume that $i_0 = 1$. Hence $P_1 |\Psi\rangle = \nu |\psi\rangle \otimes |1\rangle$, where $\nu > 0$ and $|\psi\rangle$ is a unit vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Since $P_1 \rho_{123}^n P_1 = k^{-1} \rho_{12}^n \otimes |1\rangle\langle 1|$, where

$$\rho_{12}^n = \int_0^{2\pi/n} V_x^{(1)} |\varphi_1^1\rangle\langle\varphi_1^1| (V_x^{(1)})^* \otimes V_x^{(2)} |\varphi_2^1\rangle\langle\varphi_2^1| (V_x^{(2)})^* \frac{ndx}{2\pi},$$

it follows from (16) that $\rho_{12}^n \geq k\lambda\nu|\psi\rangle\langle\psi|$. Since $P_1(\cdot)P_1$ and $\text{Tr}_K(\cdot)$ are local operations, the state $|\psi\rangle\langle\psi|$ has finite Schmidt rank. Hence Proposition 11 implies $\lambda = 0$. \square

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